

# Design and Analysis of a Novel $\mathcal{L}_1$ Adaptive Control Architecture with Guaranteed Transient Performance<sup>\*</sup>

*Chengyu Cao and Naira Hovakimyan<sup>†</sup>*

## Abstract

Conventional Model Reference Adaptive Controller (MRAC), while providing an architecture for control of systems in the presence of parametric uncertainties, offers no means for characterizing the system's input/output performance during the transient phase. Application of adaptive controllers was therefore largely restricted due to the fact that the system uncertainties during the transient have led to unpredictable/undesirebale situations, involving control signals of high-frequency or large amplitudes, large transient errors or slow convergence rate of tracking errors, to name a few. In this paper, we develop a novel adaptive control architecture that ensures that the input and the output of an uncertain linear system track the input and output of a desired linear system during the transient phase, in addition to the asymptotic tracking. This new architecture has a low-pass filter in the feedback-loop that enables to enforce the desired transient performance by increasing the adaptation gain. For the proof of asymptotic stability, the  $\mathcal{L}_1$  gain of a cascaded system, comprised of this filter and the closed-loop desired reference model, is required to be less than the inverse of the upper bound of the norm of unknown parameters used in projection based adaptation laws. The ideal (non-adaptive) version of this  $\mathcal{L}_1$  adaptive controller is used along with the main system dynamics to define a closed-loop reference system, which gives an opportunity to estimate performance bounds in terms of  $\mathcal{L}_\infty$  norms for both system's input and output signals as compared to the same signals of this reference system. Design guidelines for selection of the low-pass filter ensure that the closed-loop reference system approximates the desired system response, despite the fact that it depends upon the unknown parameters. The tools from this paper can be used to develop a theoretically justified verification and validation framework for adaptive systems. Simulation results illustrate the theoretical findings.

---

<sup>\*</sup>Research is supported by AFOSR under Contract No. FA9550-05-1-0157.

<sup>†</sup>The authors are with Aerospace & Ocean Engineering, Virginia Polytechnic Institute & State University, Blacksburg, VA 24061-0203, e-mail: chengyu, nhovakim@vt.edu

## 1 Introduction

Model Reference Adaptive Control (MRAC) architecture was developed conventionally to control linear systems in the presence of parametric uncertainties [1, 2]. The development of this architecture has been facilitated by the Lyapunov stability theory that defines sufficient conditions for stable performance, but offers no means for characterizing the system's input/output performance during the transient phase. Application of adaptive controllers was therefore largely restricted due to the fact that the system uncertainties during the transient have led to unpredictable/undesirebale situations, involving control signals of high-frequency or large amplitudes, large transient errors or slow convergence rate of tracking errors, to name a few. Extensive tuning of adaptive gains and Monte-Carlo runs have been the primary methods up today enabling the transition of adaptive control solutions to real world applications. This argument has rendered verification and validation of adaptive controllers overly challenging.

Improvement of the transient performance of adaptive controllers has been addressed from various perspectives in numerous publications [2–15], to name a few. An example presented in [12] demonstrated that the system output can have overly poor transient tracking behavior before ideal asymptotic convergence can take place. On the other hand, in [9] the author proved that it may not be possible to optimize  $\mathcal{L}_2$  and  $\mathcal{L}_\infty$  performance simultaneously by using a constant adaptation rate. Following these results, modifications of adaptive controllers were proposed in [5, 13] that render the tracking error arbitrarily small in terms of both mean-square and  $\mathcal{L}_\infty$  bounds. Further, it was shown in [3] that the modifications proposed in [5, 13] could be derived as a linear feedback of the tracking error, and the improved performance was obtained only due to a nonadaptive high-gain feedback in that scheme. In [2], composite adaptive controller was proposed, which suggests a new adaptation law using both tracking error and prediction error that leads to less oscillatory behavior in the presence of high adaptation gains as compared to MRAC. In [6], a high-gain switching MRAC technique was introduced to achieve arbitrary good transient tracking performance under a relaxed set of assumptions as compared to MRAC, and the results were shown to be of existence type only. In [15], multiple model and switching scheme is proposed to improve the transient performance of adaptive controllers. In [14], it is shown that arbitrarily close transient bound can be achieved by enforcing parameter-dependent persistent excitation condition. In [10], computable  $\mathcal{L}_2$  and  $\mathcal{L}_\infty$  bounds for the output tracking error signals are obtained for a special class of adaptive controllers using backstepping. The underlying linear nonadaptive controller possesses a parametric robustness property, however, for a large parametric uncertainty it requires high gain. In [11], dynamic certainty equivalent controllers with unnormalized es-

timators were used for adaptation that permit to derive a uniform upper bound for the  $\mathcal{L}_2$  norm of the tracking error in terms of initial parameter estimation error. In the presence of sufficiently small initial conditions, the author proved that the  $\mathcal{L}_\infty$  norm of the tracking error is upper bounded by the  $\mathcal{L}_\infty$  norm of the reference input. In [16, 17], differential game theoretic approach has been investigated for achieving arbitrarily close disturbance attenuation for tracking performance, albeit at the price of increased control effort. In [18], a new certainty equivalence based adaptive controller is presented using backstepping based control law with a normalized adaptive law to achieve asymptotic stability and guarantee performance bounds comparable with the tuning functions scheme, without the use of higher order nonlinearities.

As compared to the linear systems theory, several important aspects of the transient performance analysis seem to be missing in these papers. First, all the bounds in these papers are computed for tracking errors only, and not for control signals. Although the latter can be deduced from the former, it is straightforward to verify that the ability to adjust the former may not extend to the latter in case of nonlinear control laws. Second, since the purpose of adaptive control is to ensure stable performance in the presence of modeling uncertainties, one needs to ensure that the changes in reference input and unknown parameters due to possible faults or unexpected uncertainties do not lead to unacceptable transient deviations or oscillatory control signals, implying that a retuning of adaptive parameters is required. Finally, one needs to ensure that whatever modifications or solutions are suggested for performance improvement of adaptive controllers, they are not achieved via high-gain feedback.

Following [19], subject to appropriate trajectory initialization, the following bound  $\|e\|_\infty \leq V(t) \leq \frac{V(0)}{\lambda_{\min}(P)} \leq \frac{\tilde{\theta}^2(0)}{\Gamma}$ , where  $e$  is the tracking error,  $\tilde{\theta}$  is the parametric error,  $V(t)$  is the positive definite Lyapunov function,  $\lambda_{\min}(P)$  is the minimum eigenvalue of  $P = P^\top > 0$ , found by solving the algebraic Lyapunov equation associated with the error dynamics, implies that increasing the adaptation gain  $\Gamma$  leads to smaller tracking error for all  $t \geq 0$ , including the transient phase. However, large adaptive gain leads to high frequencies in the control signal, implying that the improvement in the transient tracking of the system output is achieved at the price of unacceptable high frequencies in the system input. One can observe from the open-loop transfer function analysis for a PI controller (which is a MRAC-structure controller for a linear system with constant disturbance) that increasing the adaptation gain leads to reduced phase-margin, and consequently reduced time-delay tolerance in input/output channels. On the contrary, decreasing the adaptive gain leads to large deviations from the desired trajectory during the transient phase.

In this paper we define a new type of model following adaptive controller that adapts fast leading to desired transient performance for the system's both input and output signals simultaneously. This new architecture has a low-pass filter in the feedback-loop that enables to enforce the desired transient performance by increasing the adaptation gain. For the proof of asymptotic stability, the  $\mathcal{L}_1$  gain of a cascaded system, comprised of this filter and the closed-loop desired transfer function, is required to be less than the inverse of the upper bound on the norm of unknown parameters used in projection based adaptation laws. With the low-pass filter in the loop, the  $\mathcal{L}_1$  adaptive controller is guaranteed to stay in the low-frequency range even in the presence of high adaptive gains and large reference inputs. The ideal (non-adaptive) version of this  $\mathcal{L}_1$  adaptive controller is used along with the main system dynamics to define a closed-loop reference system, which gives an opportunity to estimate performance bounds in terms of  $\mathcal{L}_\infty$  norms for system's both input and output signals as compared to the same signals of this reference system. These bounds immediately imply that the transient performance of the control signal in MRAC cannot be characterized. Design guidelines for selection of the low-pass filter ensure that the closed-loop reference system approximates the desired system response, despite the fact that it depends upon the unknown parameter. Thus, the desired tracking performance is achieved by systematic selection of the low-pass filter, which in its turn enables fast adaptation, as opposed to high-gain designs leading to increased control efforts. Using a simple linear system with constant disturbance, we demonstrate that this new architecture has guaranteed time-delay margin in the presence of large adaptive gain, as opposed to MRAC. We further demonstrate the extension of the methodology to systems with unknown time-varying parameters.

The paper is organized as follows. Section 2 states some preliminary definitions, and Section 3 gives the problem formulation. In Section 4, the new  $\mathcal{L}_1$  adaptive controller is presented. Stability and tracking results of the  $\mathcal{L}_1$  adaptive controller are presented in Section 5. Design guidelines are provided in Section 6. Comparison of the performance of  $\mathcal{L}_1$  adaptive controller, MRAC and the high gain controller are discussed in section 7. Analysis of  $\mathcal{L}_1$  adaptive controller in the presence of time-varying unknown parameters is presented in Section 8. In Section 9, simulation results are presented, while Section 10 concludes the paper. Unless otherwise mentioned, the  $\|\cdot\|$  will be used for the 2-norm of the vector.

## 2 Preliminaries

In this Section, we recall basic definitions and facts from linear systems theory, [8, 20, 21].

**Definition 1:** For a signal  $\xi(t), t \geq 0, \xi \in \mathbb{R}^n$ , its truncated  $\mathcal{L}_\infty$  norm and  $\mathcal{L}_\infty$  norm are  $\|\xi_t\|_{\mathcal{L}_\infty} = \max_{i=1,\dots,n} (\sup_{0 \leq \tau \leq t} |\xi_i(\tau)|)$ ,  $\|\xi\|_{\mathcal{L}_\infty} = \max_{i=1,\dots,n} (\sup_{\tau \geq 0} |\xi_i(\tau)|)$ , where  $\xi_i$  is the  $i^{th}$  component of  $\xi$ .

**Definition 2:** The  $\mathcal{L}_1$  gain of a stable proper single-input single-output system  $H(s)$  is defined to be  $\|H(s)\|_{\mathcal{L}_1} = \int_0^\infty |h(t)|dt$ .

*Proposition:* A continuous time LTI system (proper) with impulse response  $h(t)$  is stable if and only if  $\int_0^\infty |h(\tau)|d\tau < \infty$ . A proof can be found in [8] (page 81, Theorem 3.3.2).

**Definition 3:** For a stable proper  $m$  input  $n$  output system  $H(s)$  its  $\mathcal{L}_1$  gain is defined as

$$\|H(s)\|_{\mathcal{L}_1} = \max_{i=1,\dots,n} \left( \sum_{j=1}^m \|H_{ij}(s)\|_{\mathcal{L}_1} \right), \quad (1)$$

where  $H_{ij}(s)$  is the  $i^{th}$  row  $j^{th}$  column element of  $H(s)$ .

The next lemma extends the results of Example 5.2. ([20], page 199) to general multiple input multiple output systems.

**Lemma 1:** For a stable proper multi-input multi-output (MIMO) system  $H(s)$  with input  $r(t) \in \mathbb{R}^m$  and output  $x(t) \in \mathbb{R}^n$ , we have

$$\|x_t\|_{\mathcal{L}_\infty} \leq \|H\|_{\mathcal{L}_1} \|r_t\|_{\mathcal{L}_\infty}, \quad \forall t > 0. \quad (2)$$

**Corollary 1:** For a stable proper MIMO system  $H(s)$ , if the input  $r(t) \in \mathbb{R}^m$  is bounded, then the output  $x(t) \in \mathbb{R}^n$  is also bounded as  $\|x\|_{\mathcal{L}_\infty} \leq \|H(s)\|_{\mathcal{L}_1} \|r\|_{\mathcal{L}_\infty}$ .

**Lemma 2:** For a cascaded system  $H(s) = H_2(s)H_1(s)$ , where  $H_1(s)$  is a stable proper system with  $m$  inputs and  $l$  outputs and  $H_2(s)$  is a stable proper system with  $l$  inputs and  $n$  outputs, we have  $\|H(s)\|_{\mathcal{L}_1} \leq \|H_2(s)\|_{\mathcal{L}_1} \|H_1(s)\|_{\mathcal{L}_1}$ .

**Proof.** Let  $y(t) \in \mathbb{R}^n$  be the output of  $H(s) = H_1(s)H_2(s)$  in response to input  $r(t) \in \mathbb{R}^m$ . It follows from Lemma 1 that

$$\|y(t)\| \leq \|y\|_{\mathcal{L}_\infty} \leq \|H_2(s)\|_{\mathcal{L}_1} \|H_1(s)\|_{\mathcal{L}_1} \|r\|_{\mathcal{L}_\infty} \quad (3)$$

for any bounded  $r(t)$ . Let  $H_i(s), i = 1, \dots, n$  be the  $i^{th}$  row of the system  $H(s)$ . It follows from (1) that there exists  $i$  such that

$$\|H(s)\|_{\mathcal{L}_1} = \|H_i(s)\|_{\mathcal{L}_1}. \quad (4)$$

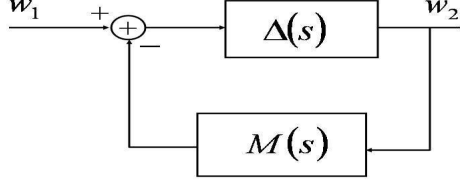


Fig. 1: Interconnected systems

Let  $h_{ij}(t)$  be the  $j^{th}$  element of the impulse response of the system  $H_i(s)$ . For any  $T$ , let

$$r_j(t) = \text{sgn} h_{ij}(T - t), \quad t \in [0, T], \quad \forall j = 1, \dots, m. \quad (5)$$

It follows from Definition 1 that  $\|r\|_{\mathcal{L}_\infty} = 1$ , and hence  $\|y(t)\| \leq \|H_2(s)\|_{\mathcal{L}_1} \|H_1(s)\|_{\mathcal{L}_1}$ ,  $\forall t \geq 0$ . For  $r(t)$  satisfying (5), we have

$$y(T) = \int_{t=0}^T \sum_{j=1}^m h_{ij}(T-t) r_j(t) dt = \int_{t=0}^T \sum_{j=1}^m |h_{ij}(T-t)| dt = \sum_{j=1}^m \left( \int_{t=0}^T |h_{ij}(t)| dt \right).$$

Therefore, it follows from (3) that for any  $T$ ,  $\sum_{j=1}^m \left( \int_{t=0}^T |h_{ij}(t)| dt \right) \leq \|H_2(s)\|_{\mathcal{L}_1} \|H_1(s)\|_{\mathcal{L}_1}$ . As  $T \rightarrow \infty$ , it follows from (4) that

$$\|H(s)\|_{\mathcal{L}_1} = \|H_i(s)\|_{\mathcal{L}_1} = \lim_{T \rightarrow \infty} \sum_{j=1}^m \left( \int_{t=0}^T |h_{ij}(t)| dt \right) \leq \|H_2(s)\|_{\mathcal{L}_1} \|H_1(s)\|_{\mathcal{L}_1},$$

and this completes the proof.  $\square$

Consider the interconnected LTI system in Fig. 1, where  $w_1 \in \mathbb{R}^{n_1}$ ,  $w_2 \in \mathbb{R}^{n_2}$ ,  $M(s)$  is a stable proper system with  $n_2$  inputs and  $n_1$  outputs, and  $\Delta(s)$  is a stable proper system with  $n_1$  inputs and  $n_2$  outputs.

**Theorem 1: ( $\mathcal{L}_1$  Small Gain Theorem)** The interconnected system in Fig. 1 is stable if  $\|M(s)\|_{\mathcal{L}_1} \|\Delta(s)\|_{\mathcal{L}_1} < 1$ .

The proof follows from Theorem 5.6 ([20], page 218), written for  $\mathcal{L}_1$  gain.

Consider a linear time invariant system:

$$\dot{x}(t) = Ax(t) + bu(t), \quad (6)$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}$ ,  $b \in \mathbb{R}^n$ ,  $A \in \mathbb{R}^{n \times n}$  is Hurwitz, and assume that the transfer function  $(sI - A)^{-1}b$  is strictly proper and stable. Notice that it can be expressed as:

$$(sI - A)^{-1}b = n(s)/d(s), \quad (7)$$

where  $d(s) = \det(sI - A)$  is a  $n^{\text{th}}$  order stable polynomial, and  $n(s)$  is a  $n \times 1$  vector with its  $i^{\text{th}}$  element being a polynomial function:

$$n_i(s) = \sum_{j=1}^n n_{ij}s^{j-1} \quad (8)$$

**Lemma 3:** If  $(A \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n)$  is controllable, the matrix  $N$  with its  $i^{\text{th}}$  row  $j^{\text{th}}$  column entry  $n_{ij}$  is full rank.

**Proof.** Controllability of  $(A, b)$  for the LTI system in (6) implies that given an initial condition  $x(t_0) = 0$  and arbitrary  $x_{t_1} \in \mathbb{R}^n$  and arbitrary  $t_1$ , there exists  $u(\tau), \tau \in [t_0, t_1]$  such that  $x(t_1) = x_{t_1}$ . If  $N$  is not full rank, then there exists a non-zero vector  $u \in \mathbb{R}^n$ , such that  $u^\top n(s) = 0$ . Then it follows that for  $x(t_0) = 0$  one has  $u^\top(\tau)x(\tau) = 0, \forall \tau > t_0$ . This contradicts  $x(t_1) = x_{t_1}$ , in which  $x_{t_1} \in \mathbb{R}^n$  is assumed to be an arbitrary point. Therefore,  $N$  must be full rank, and the proof is complete.  $\square$

**Lemma 4:** If  $(A, b)$  is controllable and  $(sI - A)^{-1}b$  is strictly proper and stable, there exists  $c \in \mathbb{R}^n$  such that the transfer function  $c^\top(sI - A)^{-1}b$  is minimum phase with relative degree one, i.e. all its zeros are located in the left half plane, and its denominator is one order larger than its numerator.

**Proof.** It follows from (7) that

$$c^\top(sI - A)^{-1}b = (c^\top N[s^{n-1} \dots 1]^\top)/d(s), \quad (9)$$

where  $N \in \mathbb{R}^{n \times n}$  is matrix with its  $i^{\text{th}}$  row  $j^{\text{th}}$  column entry  $n_{ij}$  introduced in (8). We choose  $\bar{c} \in \mathbb{R}^n$  such that  $\bar{c}^\top[s^{n-1} \dots 1]^\top$  is a stable  $n - 1$  order polynomial. Since  $(A, b)$  is controllable, it follows from Lemma 3 that  $N$  is full rank. Let  $c = (N^{-1})^\top \bar{c}$ . Then it follows from (9) that  $c^\top(sI - A)^{-1}b = \frac{\bar{c}^\top[s^{n-1} \dots 1]^\top}{d(s)}$  has relative degree 1 with all its zeros in the left half plane.  $\square$

### 3 Problem Formulation

Consider the following single-input single-output system dynamics:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + b(u(t) - \theta^\top x(t)), & x(0) &= x_0 \\ y(t) &= c^\top x(t), \end{aligned} \quad (10)$$

where  $x \in \mathbb{R}^n$  is the system state vector (measurable),  $u \in \mathbb{R}$  is the control signal,  $b, c \in \mathbb{R}^n$  are known constant vectors,  $A$  is known  $n \times n$  matrix,  $(A, b)$  is controllable, unknown parameter  $\theta \in \mathbb{R}^n$  belongs to a given compact convex set  $\theta \in \Omega$ ,  $y \in \mathbb{R}$  is the regulated output.

The control objective is to design a low-frequency adaptive controller to ensure that  $y(t)$  tracks a given bounded continuous reference signal  $r(t)$  *both in transient and steady state*, while all other error signals remain bounded. More rigorously, the control objective can be stated as design of a low-pass control signal  $u(t)$  to achieve

$$y(s) \approx D(s)r(s), \quad (11)$$

where  $y(s), r(s)$  are Laplace transformations of  $y(t), r(t)$  respectively, and  $D(s)$  is a strictly proper stable LTI system that specifies the desired transient and steady state performance. We note that the control objective can be met if both the control signal  $u(t)$  and the system response  $x(t)$  approximate the corresponding signals of a LTI system with its response close to  $D(s)$ .

## 4 $\mathcal{L}_1$ Adaptive Controller

In this section, we develop a novel adaptive control architecture that permits complete transient characterization for system's both input and output signals. Since  $(A, b)$  is controllable, we choose  $K$  to ensure that  $A_m = A - bK^\top$  is Hurwitz or, equivalently, that

$$H_o(s) = (sI - A_m)^{-1}b \quad (12)$$

is stable. The following control structure

$$u(t) = u_1(t) + u_2(t), \quad u_1(t) = -K^\top x(t), \quad (13)$$

where  $u_2(t)$  is the adaptive controller to be determined later, leads to the following *partially* closed-loop dynamics:

$$\begin{aligned} \dot{x}(t) &= A_m x(t) - b\theta^\top x(t) + bu_2(t), \quad x(0) = x_0 \\ y(t) &= c^\top x(t). \end{aligned} \quad (14)$$

For the linearly parameterized system in (14), we consider the following state predictor

$$\begin{aligned} \dot{\hat{x}}(t) &= A_m \hat{x}(t) - b\hat{\theta}^\top(t)x(t) + bu_2(t), \quad \hat{x}(0) = x_0 \\ \hat{y}(t) &= c^\top \hat{x}(t) \end{aligned} \quad (15)$$

along with the adaptive law for  $\hat{\theta}(t)$ :

$$\dot{\hat{\theta}}(t) = \Gamma \text{Proj}(\hat{\theta}(t), x(t)\tilde{x}^\top(t)Pb), \quad \hat{\theta}(0) = \hat{\theta}_0, \quad (16)$$

where  $\tilde{x}(t) = \hat{x}(t) - x(t)$  is the prediction error,  $\Gamma \in \mathbb{R}^{n \times n} = \Gamma_c I_{n \times n}$  is the matrix of adaptation gains, and  $P$  is the solution of the algebraic equation  $A_m^\top P + P A_m = -Q$ ,  $Q > 0$ .

Letting

$$\bar{r}(t) = \hat{\theta}^\top(t)x(t), \quad (17)$$

the state predictor in (15) can be viewed as a low-pass system with  $u_2(t)$  being its control signal,  $\bar{r}(t)$  being a time-varying disturbance, which is not prevented from having high-frequency oscillations. We consider the following control design for (15):

$$u_2(s) = C(s)(\bar{r}(s) + k_g r(s)), \quad (18)$$

where  $u_2(s)$ ,  $\bar{r}(s)$ ,  $r(s)$  are the Laplace transformations of  $u_2(t)$ ,  $\bar{r}(t)$ ,  $r(t)$ , respectively,  $C(s)$  is a stable and strictly proper system with low-pass gain  $C(0) = 1$ , and  $k_g$  is

$$k_g = \lim_{s \rightarrow 0} \frac{1}{c^\top H_o(s)} = \frac{1}{c^\top H_o(0)}. \quad (19)$$

The complete  $\mathcal{L}_1$  adaptive controller consists of (13), (15), (16), (18), and the closed-loop system with it is illustrated in Fig. 2.

Fig. 2: Closed-loop system with  $\mathcal{L}_1$  adaptive controller

Consider the closed-loop state predictor in (15) with the control signal defined in (18). It can be viewed as an LTI system with two inputs  $r(t)$  and  $\bar{r}(t)$ :

$$\hat{x}(s) = \bar{G}(s)\bar{r}(s) + G(s)r(s) \quad (20)$$

$$\bar{G}(s) = H_o(s)(C(s) - 1) \quad (21)$$

$$G(s) = k_g H_o(s)C(s), \quad (22)$$

where  $\hat{x}(s)$  is the Laplace transformation of  $\hat{x}(t)$ . We note that  $\bar{r}(t)$  is related to  $\hat{x}(t)$ ,  $u(t)$  and  $r(t)$  via nonlinear relationships.

**Remark 1:** Since both  $H_o(s)$  and  $C(s)$  are strictly proper stable systems, one can check easily that  $\bar{G}(s)$  and  $G(s)$  are strictly proper stable systems, even though that  $1 - C(s)$  is proper.

Let

$$\theta_{\max} = \max_{\theta \in \Omega} \sum_{i=1}^n |\theta_i|, \quad (23)$$

where  $\theta_i$  is the  $i^{th}$  element of  $\theta$ ,  $\Omega$  is the compact set, where the unknown parameter lies. We now give the  $\mathcal{L}_1$  performance requirement that ensures stability of the entire system and desired transient performance, as discussed later in Section 5.

**$\mathcal{L}_1$ -gain requirement:** Design  $K$  and  $C(s)$  to satisfy

$$\|\bar{G}(s)\|_{\mathcal{L}_1} \theta_{\max} < 1. \quad (24)$$

## 5 Analysis of $\mathcal{L}_1$ Adaptive Controller

### 5.1 Stability and Asymptotic Convergence

Consider the following Lyapunov function candidate:

$$V(\tilde{x}(t), \tilde{\theta}(t)) = \tilde{x}^\top(t)P\tilde{x}(t) + \tilde{\theta}^\top(t)\Gamma^{-1}\tilde{\theta}(t), \quad (25)$$

where  $P$  and  $\Gamma$  are introduced in (16). It follows from (14) and (15) that

$$\dot{\tilde{x}}(t) = A_m \tilde{x}(t) - b \tilde{\theta}^\top(t)x(t), \quad \tilde{x}(0) = 0. \quad (26)$$

Hence, it is straightforward to verify from (16) that

$$\dot{V}(t) \leq -\tilde{x}^\top(t)Q\tilde{x}(t) \leq 0. \quad (27)$$

Notice that the result in (27) is independent of  $u_2(t)$ , however, one cannot deduce stability from it. One needs to prove in addition that with the  $\mathcal{L}_1$  adaptive controller the state of the predictor will remain bounded. Boundedness of the system state then will follow.

**Theorem 2:** Given the system in (10) and the  $\mathcal{L}_1$  adaptive controller defined via (13), (15), (16), (18) subject to (24), the tracking error  $\tilde{x}(t)$  converges to zero asymptotically:

$$\lim_{t \rightarrow \infty} \tilde{x}(t) = 0. \quad (28)$$

**Proof.** Let  $\lambda_{\min}(P)$  be the minimum eigenvalue of  $P$ . From (25) and (27) it follows that  $\lambda_{\min}(P)\|\tilde{x}(t)\|^2 \leq \tilde{x}^\top(t)P\tilde{x}(t) \leq V(t) \leq V(0)$ , implying that

$$\|\tilde{x}(t)\|^2 \leq V(0)/\lambda_{\min}(P), \quad t \geq 0. \quad (29)$$

From Definition 1,  $\|\tilde{x}\|_{\mathcal{L}_\infty} = \max_{i=1,\dots,n,t \geq 0} |\tilde{x}_i(t)|$ . The relationship in (29) ensures

$$\max_{i=1,\dots,n,t \geq 0} |\tilde{x}_i(t)| \leq \sqrt{V(0)/\lambda_{\min}(P)}, \text{ and therefore for all } t > 0 \text{ one has } \|\tilde{x}_t\|_{\mathcal{L}_\infty} \leq \sqrt{V(0)/\lambda_{\min}(P)}.$$

Using the triangular relationship for norms implies that

$$|\|\hat{x}_t\|_{\mathcal{L}_\infty} - \|x_t\|_{\mathcal{L}_\infty}| \leq \sqrt{V(0)/\lambda_{\min}(P)}. \quad (30)$$

The projection algorithm in (16) ensures that  $\hat{\theta}(t) \in \Omega, \forall t \geq 0$ . The definition of  $\bar{r}(t)$  in (17) implies that  $\|\bar{r}_t\|_{\mathcal{L}_\infty} \leq \theta_{\max}\|x_t\|_{\mathcal{L}_\infty}$ . Substituting for  $\|x_t\|_{\mathcal{L}_\infty}$  from (30) leads to the following

$$\|\bar{r}_t\|_{\mathcal{L}_\infty} \leq \theta_{\max} \left( \|\hat{x}_t\|_{\mathcal{L}_\infty} + \sqrt{V(0)/\lambda_{\min}(P)} \right). \quad (31)$$

It follows from Lemma 1 that  $\|\hat{x}_t\|_{\mathcal{L}_\infty} \leq \|\bar{G}(s)\|_{\mathcal{L}_1}\|\bar{r}_t\|_{\mathcal{L}_\infty} + \|G(s)\|_{\mathcal{L}_1}\|r_t\|_{\mathcal{L}_\infty}$ , which along with (31) gives the following upper bound

$$\|\hat{x}_t\|_{\mathcal{L}_\infty} \leq \|\bar{G}(s)\|_{\mathcal{L}_1}\theta_{\max} \left( \|\hat{x}_t\|_{\mathcal{L}_\infty} + \sqrt{V(0)/\lambda_{\min}(P)} \right) + \|G(s)\|_{\mathcal{L}_1}\|r_t\|_{\mathcal{L}_\infty}. \quad (32)$$

Let

$$\lambda = \|\bar{G}(s)\|_{\mathcal{L}_1}\theta_{\max}. \quad (33)$$

From (24) it follows that  $\lambda < 1$ . The relationship in (32) can be written as  $(1 - \lambda)\|\hat{x}_t\|_{\mathcal{L}_\infty} \leq \lambda\sqrt{V(0)/\lambda_{\min}(P)} + \|G(s)\|_{\mathcal{L}_1}\|r_t\|_{\mathcal{L}_\infty}$ , and hence

$$\|\hat{x}_t\|_{\mathcal{L}_\infty} \leq (\lambda\sqrt{V(0)/\lambda_{\min}(P)} + \|G(s)\|_{\mathcal{L}_1}\|r_t\|_{\mathcal{L}_\infty})/(1 - \lambda). \quad (34)$$

Since  $V(0)$ ,  $\lambda_{\min}(P)$ ,  $\|G(s)\|_{\mathcal{L}_1}$ ,  $\|r\|_{\mathcal{L}_\infty}$ ,  $\lambda$  are all finite and  $\lambda < 1$ , the relationship in (34) implies that  $\|\hat{x}_t\|_{\mathcal{L}_\infty}$  is finite for any  $t > 0$ , and hence  $\hat{x}(t)$  is bounded. The relationship in (30) implies that  $\|x_t\|_{\mathcal{L}_\infty}$  is also finite for all  $t > 0$ , and therefore  $x(t)$  is bounded. The adaptive law in (16) ensures that the estimates  $\hat{\theta}(t)$  are also bounded. From (26) it follows that  $\dot{\tilde{x}}(t)$  is bounded, and it follows from Barbalat's lemma that  $\lim_{t \rightarrow \infty} \tilde{x}(t) = 0$ .  $\square$

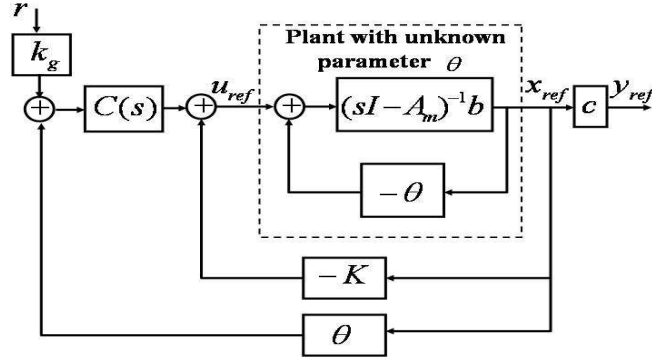


Fig. 3: Closed-loop reference LTI system

## 5.2 Reference System

In this section we characterize the reference system that the  $\mathcal{L}_1$  adaptive controller in (13), (15), (16), (18) tracks both in transient and steady state, and this tracking is valid for system's both input and output signals. Towards that end, consider the following *ideal* version of the adaptive controller in (13), (18):

$$u_{ref}(s) = C(s) (k_g r(s) + \eta(s)) - K^\top x_{ref}(s), \quad (35)$$

where  $\eta(s)$  is the Laplace transformation of

$$\eta(t) = \theta^\top x_{ref}(t), \quad (36)$$

and  $x_{ref}(s)$  is used to denote the Laplace transformation of the state  $x_{ref}(t)$  of the closed-loop system. The closed-loop system (10) with the controller (35) is given in Fig. 3.

**Remark 2:** Notice that when  $C(s) = 1$ , one recovers the reference model of MRAC. If  $C(s) \neq 1$ , then the control law in (35) changes the bandwidth of the ideal control signal  $u_{ideal}(t) = -K^\top x(t) + \theta^\top x(t) + k_g r(t)$ .

The control law in (35) leads to the following closed-loop dynamics:

$$\begin{aligned} x_{ref}(s) &= H_o(s) \left( k_g C(s) r(s) + (C(s) - 1) \theta^\top x_{ref}(s) \right) \\ y_{ref}(s) &= c^\top x_{ref}(s), \end{aligned} \quad (37)$$

which can be explicitly solved for  $x_{ref}(s) = \left(I - (C(s) - 1)H_o(s)\theta^\top\right)^{-1} H_o(s)k_g C(s)r(s)$ . Hence, it follows from (21) and (22) that

$$x_{ref}(s) = (I - \bar{G}(s)\theta^\top)^{-1}G(s)r(s). \quad (38)$$

**Lemma 5:** If  $\|\bar{G}(s)\|_{\mathcal{L}_1}\theta_{\max} < 1$ , then

$$\begin{aligned} (i) \quad & (I - \bar{G}(s)\theta^\top)^{-1} \text{ is stable;} \\ (ii) \quad & (I - \bar{G}(s)\theta^\top)^{-1}G(s) \text{ is stable.} \end{aligned} \quad (39)$$

**Proof.** It follows from (1) that

$$\|\bar{G}(s)\theta^\top\|_{\mathcal{L}_1} = \max_{i=1,\dots,n} \left( \|\bar{G}_i(s)\|_{\mathcal{L}_1} \left( \sum_{j=1}^n |\theta_j| \right) \right),$$

where  $\bar{G}_i(s)$  is the  $i^{th}$  element of  $G(s)$ , and  $\theta_j$  is the  $j^{th}$  element of  $\theta$ . From (23) we have  $\sum_{j=1}^n |\theta_j| \leq \theta_{\max}$ , and hence

$$\|\bar{G}(s)\theta^\top\|_{\mathcal{L}_1} \leq \max_{i=1,\dots,n} \left( \|\bar{G}_i(s)\|_{\mathcal{L}_1} \right) \theta_{\max} = \|\bar{G}(s)\|_{\mathcal{L}_1} \theta_{\max}, \quad \forall \theta \in \Omega. \quad (40)$$

The relationship in (24) implies that  $\|\bar{G}(s)\theta^\top\|_{\mathcal{L}_1} < 1$ , and therefore Theorem 1 ensures that the LTI system  $(I - \bar{G}(s)\theta^\top)^{-1}$  is stable. Since  $G(s)$  is stable, Remark 1 implies that  $(I - \bar{G}(s)\theta^\top)^{-1}G(s)$  is stable.  $\square$

### 5.3 System Response and Control Signal of the $\mathcal{L}_1$ Adaptive Controller

Letting

$$r_1(t) = \tilde{\theta}^\top(t)x(t), \quad (41)$$

we notice that  $\bar{r}(t)$  in (17) can be rewritten as  $\bar{r}(t) = \theta^\top(\hat{x}(t) - \tilde{x}(t)) + r_1(t)$ . Hence, the state predictor in (20) can be rewritten as  $\hat{x}(s) = \bar{G}(s) (\theta^\top \hat{x}(s) - \theta^\top \tilde{x}(s) + r_1(s)) + G(s)r(s)$ , where  $r_1(s)$  is the Laplace transformation of  $r_1(t)$  defined in (41), and further put into the form:

$$\hat{x}(s) = (I - \bar{G}(s)\theta^\top)^{-1} \left( -\bar{G}(s)\theta^\top \tilde{x}(s) + \bar{G}(s)r_1(s) + G(s)r(s) \right). \quad (42)$$

It follows from (26) and (41) that  $\dot{\tilde{x}}(t) = A_m \tilde{x}(t) - b r_1(t)$ , and hence

$$\tilde{x}(s) = -H_o(s)r_1(s). \quad (43)$$

Using the expression of  $\bar{G}(s)$  from (21), the state of the predictor can be presented as  $\hat{x}(s) = (I - \bar{G}(s)\theta^\top)^{-1}G(s)r(s) + (I - \bar{G}(s)\theta^\top)^{-1}(-\bar{G}(s)\theta^\top \tilde{x}(s) - (C(s) - 1)\tilde{x}(s))$ . Using  $x_{ref}(s)$  from (38) and recalling the definition of  $\tilde{x}(s) = \hat{x}(s) - x(s)$ , one arrives at

$$x(s) = x_{ref}(s) - \left( I + (I - \bar{G}(s)\theta^\top)^{-1}(\bar{G}(s)\theta^\top + (C(s) - 1)I) \right) \tilde{x}(s). \quad (44)$$

The expressions in (13), (18) and (35) lead to the following expression of the control signal

$$u(s) = u_{ref}(s) + C(s)r_1(s) + (C(s)\theta^\top - K^\top)(x(s) - x_{ref}(s)). \quad (45)$$

#### 5.4 Asymptotic Performance and Steady State Error

**Theorem 3:** Given the system in (10) and the  $\mathcal{L}_1$  adaptive controller defined via (13), (15), (16), (18) subject to (24), we have:

$$\lim_{t \rightarrow \infty} \|x(t) - x_{ref}(t)\| = 0, \quad (46)$$

$$\lim_{t \rightarrow \infty} |u(t) - u_{ref}(t)| = 0. \quad (47)$$

**Proof.** Let

$$r_2(t) = x_{ref}(t) - x(t). \quad (48)$$

It follows from (44) that

$$r_2(s) = \left( I + (I - \bar{G}(s)\theta^\top)^{-1} \left( \bar{G}(s)\theta^\top + (C(s) - 1)I \right) \right) \tilde{x}(s). \quad (49)$$

The signal  $r_2(t)$  can be viewed as the response of the LTI system

$$H_2(s) = I + (I - \bar{G}(s)\theta^\top)^{-1} \left( \bar{G}(s)\theta^\top + (C(s) - 1)I \right) \quad (50)$$

to the bounded error signal  $\tilde{x}(t)$ . It follows from (39) and Remark 1 that  $(I - \bar{G}(s)\theta^\top)^{-1}$ ,  $\bar{G}(s)$ ,  $C(s)$  are stable and, therefore,  $H_2(s)$  is stable. Hence, from (28) we have  $\lim_{t \rightarrow \infty} r_2(t) = 0$ . Let

$$r_3(s) = C(s)r_1(s) + (C(s)\theta^\top - K^\top)(x(s) - x_{ref}(s)). \quad (51)$$

It follows from (45) that

$$r_3(t) = u(t) - u_{ref}(t). \quad (52)$$

Since the projection operator ensures that  $\tilde{\theta}(t)$  is bounded, it follows from (26) and (28) that  $\lim_{t \rightarrow \infty} r_1(t) = 0$ . Since  $C(s)$  is a stable proper system, it follows from (46) that  $\lim_{t \rightarrow \infty} r_3(t) = 0$ .  $\square$

**Lemma 6:** Given the system in (10) and the  $\mathcal{L}_1$  adaptive controller defined via (13), (15), (16), (18) subject to (24), if  $r(t)$  is constant, then  $\lim_{t \rightarrow \infty} y(t) = r$ .

**Proof.** Since  $y_{ref}(t) = c^\top x_{ref}(t)$ , it follows from (46) that  $\lim_{t \rightarrow \infty} (y(t) - y_{ref}(t)) = 0$ . From (38) it follows that  $y_{ref}(s) = c^\top (I - \bar{G}(s)\theta^\top)^{-1} G(s)r(s)$ . The end value theorem ensures

$$\lim_{t \rightarrow \infty} y_{ref}(t) = \lim_{s \rightarrow 0} c^\top (I - \bar{G}(s)\theta^\top)^{-1} G(s)r = c^\top H_o(0)C(0)k_g r. \quad (53)$$

Definition of  $k_g$  in (19) leads to  $\lim_{t \rightarrow \infty} y(t) = r$ .  $\square$

## 5.5 Bounded Tracking Error Signal

**Lemma 7:** Let  $\Gamma = \Gamma_c \mathbb{I}$ , where  $\Gamma_c \in \mathbb{R}^+$ , and  $\mathbb{I}$  is the identity matrix. For the system in (10)

$$\|\tilde{x}(t)\| \leq \sqrt{\bar{\theta}_{\max}/(\lambda_{\min}(P)\Gamma_c)}, \quad \bar{\theta}_{\max} \triangleq \max_{\theta \in \Omega} \sum_{i=1}^n 4\theta_i^2, \quad \forall t \geq 0, \quad (54)$$

and  $\lambda_{\min}(P)$  is the minimum eigenvalue of  $P$ .

**Proof.** For the  $V(t)$  in (25), the following upper bound is straightforward to derive:  $\tilde{x}^\top(t)P\tilde{x}(t) \leq V(t) \leq V(0), \forall t \geq 0$ . The projection algorithm ensures that  $\hat{\theta}(t) \in \Omega, \forall t \geq 0$ , and therefore

$$\max_{t \geq 0} \tilde{\theta}^\top(t)\Gamma^{-1}\tilde{\theta}(t) \leq \frac{\bar{\theta}_{\max}}{\Gamma_c}, \quad \forall t \geq 0, \quad (55)$$

where  $\bar{\theta}_{\max}$  is defined in (54). Since  $\tilde{x}(0) = 0$ , then  $V(0) = \tilde{\theta}^\top(0)\Gamma^{-1}\tilde{\theta}(0)$ , which leads to  $\tilde{x}^\top(t)P\tilde{x}(t) \leq \frac{\bar{\theta}_{\max}}{\Gamma_c}, t \geq 0$ . Since  $\lambda_{\min}(P)\|\tilde{x}\|^2 \leq \tilde{x}^\top(t)P\tilde{x}(t)$ ,

$$\text{then } \|\tilde{x}(t)\| \leq \sqrt{\frac{\bar{\theta}_{\max}}{\lambda_{\min}(P)\Gamma_c}}.$$

## 5.6 Transient Performance

We note that  $(A - bK^\top, b)$  is the state space realization of  $H_o(s)$ . Since  $(A, b)$  is controllable, it can be easily proved that  $(A - bK^\top, b)$  is also controllable. It follows from Lemma 4 that there exists  $c_o \in \mathbb{R}^n$  such that

$$c_o^\top H_o(s) = N_n(s)/N_d(s), \quad (56)$$

where the order of  $N_d(s)$  is one more than the order of  $N_n(s)$ , and both  $N_n(s)$  and  $N_d(s)$  are stable polynomials.

**Theorem 4:** Given the system in (10) and the  $\mathcal{L}_1$  adaptive controller defined via (13), (15), (16), (18) subject to (24), we have:

$$\|x - x_{ref}\|_{\mathcal{L}_\infty} \leq \gamma_1 / \sqrt{\Gamma_c}, \quad (57)$$

$$\|y - y_{ref}\|_{\mathcal{L}_\infty} \leq \|c^\top\|_{\mathcal{L}_1} \gamma_1 / \sqrt{\Gamma_c}, \quad (58)$$

$$\|u - u_{ref}\|_{\mathcal{L}_\infty} \leq \gamma_2 / \sqrt{\Gamma_c}, \quad (59)$$

where  $\|c^\top\|_{\mathcal{L}_1}$  is the  $\mathcal{L}_1$  gain of  $c^\top$  and

$$\gamma_1 = \|H_2(s)\|_{\mathcal{L}_1} \sqrt{\bar{\theta}_{\max} / \lambda_{\max}(P)}, \quad (60)$$

$$\gamma_2 = \left\| C(s) \frac{1}{c_o^\top H_o(s)} c_o^\top \right\|_{\mathcal{L}_1} \sqrt{\bar{\theta}_{\max} / \lambda_{\max}(P)} + \|C(s)\theta^\top - K^\top\|_{\mathcal{L}_1} \gamma_1 \quad (61)$$

**Proof.** It follows from (49), (50) and Lemma 1 that  $\|r_2\|_{\mathcal{L}_\infty} \leq \|H_2(s)\|_{\mathcal{L}_1} \|\tilde{x}\|_{\mathcal{L}_\infty}$ , while Lemma 7 implies that

$$\|\tilde{x}\|_{\mathcal{L}_\infty} \leq \sqrt{\bar{\theta}_{\max} / (\lambda_{\max}(P) \Gamma_c)}. \quad (62)$$

Therefore,  $\|r_2\|_{\mathcal{L}_\infty} \leq \|H_2(s)\|_{\mathcal{L}_1} \sqrt{\frac{\bar{\theta}_{\max}}{\lambda_{\max}(P) \Gamma_c}}$ , which leads to (57). The upper bound in (58) follows from (57) and Lemma 2 directly. From (43) we have

$$\begin{aligned} r_3(s) &= C(s) \frac{1}{c_o^\top H_o(s)} c_o^\top H_o(s) r_1(s) + (C(s)\theta^\top - K^\top)(x(s) - x_{ref}(s)) \\ &= -C(s) \frac{1}{c_o^\top H_o(s)} c_o^\top \tilde{x}(s) + (C(s)\theta^\top - K^\top)(x(s) - x_{ref}(s)), \end{aligned}$$

where  $c_o$  is introduced in (56). It follows from (56) that  $C(s) \frac{1}{c_o^\top H_o(s)} = C(s) \frac{N_d(s)}{N_n(s)}$ , where  $N_d(s)$ ,  $N_n(s)$  are stable polynomials and the order of  $N_n(s)$  is one less than the order of  $N_d(s)$ . Since  $C(s)$  is stable and strictly proper, the complete system  $C(s) \frac{1}{c_o^\top H_o(s)}$  is proper and stable, which implies that its  $\mathcal{L}_1$  gain exists and is finite.

Hence, we have  $\|r_3\|_{\mathcal{L}_\infty} \leq \left\| C(s) \frac{1}{c_o^\top H_o(s)} c_o^\top \right\|_{\mathcal{L}_1} \|\tilde{x}\|_{\mathcal{L}_\infty} + \|C(s)\theta^\top - K^\top\|_{\mathcal{L}_1} \|x - x_{ref}\|_{\mathcal{L}_\infty}$ . Lemma 7 leads to the upper bound in (59):

$$\|r_3\|_{\mathcal{L}_\infty} \leq \left\| C(s) \frac{1}{c_o^\top H_o(s)} c_o^\top \right\|_{\mathcal{L}_1} \sqrt{\frac{\bar{\theta}_{\max}}{\lambda_{\max}(P) \Gamma_c}} + \|C(s)\theta^\top - K^\top\|_{\mathcal{L}_1} \|x - x_{ref}\|_{\mathcal{L}_\infty}.$$

□

**Corollary 2:** Given the system in (10) and the  $\mathcal{L}_1$  adaptive controller defined via (13), (15), (16), (18) subject to (24), we have:

$$\lim_{\Gamma_c \rightarrow \infty} (x(t) - x_{ref}(t)) = 0, \quad \forall t \geq 0, \quad (63)$$

$$\lim_{\Gamma_c \rightarrow \infty} (y(t) - y_{ref}(t)) = 0, \quad \forall t \geq 0, \quad (64)$$

$$\lim_{\Gamma_c \rightarrow \infty} (u(t) - u_{ref}(t)) = 0, \quad \forall t \geq 0. \quad (65)$$

Corollary 2 states that  $x(t)$ ,  $y(t)$  and  $u(t)$  follow  $x_{ref}(t)$ ,  $y_{ref}(t)$  and  $u_{ref}(t)$  not only asymptotically but also during the transient, provided that the adaptive gain is selected sufficiently large. Thus, the control objective is reduced to designing  $K$  and  $C(s)$  to ensure that the reference LTI system has the desired response  $D(s)$ .

**Remark 3:** Notice that if we set  $C(s) = 1$ , then the  $\mathcal{L}_1$  adaptive controller degenerates into a MRAC type. In that case  $\left\| C(s) \frac{1}{c_o^\top H_o(s)} c_o^\top \right\|_{\mathcal{L}_1}$  cannot be finite, since  $H_o(s)$  is strictly proper. Therefore, from (61) it follows that  $\gamma_2 \rightarrow \infty$ , and hence for the control signal in MRAC one can not reduce the bound in (59) by increasing the adaptive gain.

## 6 Design of the $\mathcal{L}_1$ Adaptive Controller

We proved that the error between the state and the control signal of the closed-loop system with  $\mathcal{L}_1$  adaptive controller in (10), (13), (15), (16), (18) (Fig. 2) and the state and the control signal of the closed-loop reference system in (35), (38) (Fig. 3) can be rendered arbitrarily small by choosing large adaptive gain. Therefore, the control objective is reduced to determining  $K$  and  $C(s)$  to ensure that the reference system in (35), (38) (Fig. 3) has the desired response  $D(s)$  from  $r(t)$  to  $y_{ref}(t)$ . Notice that the reference system in Fig. 3 depends upon the unknown parameter  $\theta$ .

Consider the following signals:

$$y_{des}(s) = c^\top G(s) r(s) = C(s) k_g c^\top H_o(s) r(s), \quad (66)$$

$$u_{des}(s) = k_g C(s) \left( 1 + C(s) \theta^\top H_o(s) - K^\top H_o(s) \right) r(s). \quad (67)$$

We note that  $u_{des}(t)$  depends on the unknown parameter  $\theta$ , while  $y_{des}(t)$  does not.

**Lemma 8:** Subject to (24), the following upper bounds hold:

$$\|y_{ref} - y_{des}\|_{\mathcal{L}_\infty} \leq \frac{\lambda}{1 - \lambda} \|c^\top\|_{\mathcal{L}_1} \|G(s)\|_{\mathcal{L}_1} \|r\|_{\mathcal{L}_\infty}, \quad (68)$$

$$\|y_{ref} - y_{des}\|_{\mathcal{L}_\infty} \leq \frac{1}{1-\lambda} \|c^\top\|_{\mathcal{L}_1} \|h_3\|_{\mathcal{L}_\infty}, \quad (69)$$

$$\|u_{ref} - u_{des}\|_{\mathcal{L}_\infty} \leq \frac{\lambda}{1-\lambda} \|C(s)\theta^\top - K^\top\|_{\mathcal{L}_1} \|G(s)\|_{\mathcal{L}_1} \|r\|_{\mathcal{L}_\infty}, \quad (70)$$

$$\|u_{ref} - u_{des}\|_{\mathcal{L}_\infty} \leq \frac{1}{1-\lambda} \|C(s)\theta^\top - K^\top\|_{\mathcal{L}_1} \|h_3\|_{\mathcal{L}_\infty}, \quad (71)$$

where  $\lambda$  is defined in (33), and  $h_3(t)$  is the inverse Laplace transformation of

$$H_3(s) = (C(s) - 1)C(s)r(s)k_g H_o(s)\theta^\top H_o(s). \quad (72)$$

**Proof.** It follows from (37) and (38) that  $y_{ref}(s) = c^\top (I - \bar{G}(s)\theta^\top)^{-1} G(s)r(s)$ . Following Lemma 5, the condition in (24) ensures the stability of the reference LTI system. Since  $(I - \bar{G}(s)\theta^\top)^{-1}$  is stable, then one can expand it into convergent series and further write

$$y_{ref}(s) = c^\top \left( I + \sum_{i=1}^{\infty} (\bar{G}(s)\theta^\top)^i \right) G(s)r(s) = y_{des}(s) + c^\top \left( \sum_{i=1}^{\infty} (\bar{G}(s)\theta^\top)^i \right) G(s)r(s). \quad (73)$$

Let  $r_4(s) = c^\top \left( \sum_{i=1}^{\infty} (\bar{G}(s)\theta^\top)^i \right) G(s)r(s)$ . Then

$$r_4(t) = y_{ref}(t) - y_{des}(t), \quad \forall t \geq 0. \quad (74)$$

The relationship in (40) implies that  $\|\bar{G}(s)\theta^\top\|_{\mathcal{L}_1} \leq \lambda$ , and it follows from Lemma 2 that

$$\|r_4\|_{\mathcal{L}_\infty} \leq \left( \sum_{i=1}^{\infty} \lambda^i \right) \|c^\top\|_{\mathcal{L}_1} \|G\|_{\mathcal{L}_1} \|r\|_{\mathcal{L}_\infty} = \frac{\lambda}{1-\lambda} \|c^\top\|_{\mathcal{L}_1} \|G\|_{\mathcal{L}_1} \|r\|_{\mathcal{L}_\infty}. \quad (75)$$

From (73) we have  $y_{ref}(s) = y_{des}(s) + c^\top \left( \sum_{i=1}^{\infty} (\bar{G}(s)\theta^\top)^{i-1} \right) \bar{G}(s)\theta^\top G(s)r(s)$ , which along with (21), (22) and (72) leads to

$$y_{ref}(s) = y_{des}(s) + c^\top \left( \sum_{i=1}^{\infty} (\bar{G}(s)\theta^\top)^{i-1} \right) H_3(s).$$

Lemma 1 immediately implies that  $\|r_4\|_{\mathcal{L}_\infty} \leq \left( \sum_{i=1}^{\infty} \lambda^{i-1} \right) \|c^\top\|_{\mathcal{L}_1} \|h_3\|_{\mathcal{L}_\infty}$ . Comparing  $u_{des}(s)$  in (67) to  $u_{ref}(s)$  in (35) it follows that  $u_{des}(s)$  can be written as  $u_{des}(s) = k_g C(s)r(s) + (C(s)\theta^\top - K^\top)x_{des}(s)$ , where  $x_{des}(s) = C(s)k_g H_o(s)r(s)$ . Therefore  $u_{ref}(s) - u_{des}(s) = (C(s)\theta^\top - K^\top)(x_{ref}(s) - x_{des}(s))$ . Hence, it follows from Lemma 1 that  $\|u_{ref} - u_{des}\|_{\mathcal{L}_\infty} \leq \|C(s)\theta^\top - K^\top\|_{\mathcal{L}_1} \|x_{ref} - x_{des}\|_{\mathcal{L}_\infty}$ .

Using the same steps as for  $\|y_{ref} - y_{des}\|_{\mathcal{L}_\infty}$ , we have

$$\begin{aligned}\|x_{ref} - x_{des}\|_{\mathcal{L}_\infty} &\leq \frac{\lambda}{1-\lambda} \|G(s)\|_{\mathcal{L}_1} \|r\|_{\mathcal{L}_\infty}, \\ \|x_{ref} - x_{des}\|_{\mathcal{L}_\infty} &\leq \frac{1}{1-\lambda} \|h_3\|_{\mathcal{L}_\infty},\end{aligned}$$

which leads to the upper bounds in (70) and (71).  $\square$

Thus, the problem is reduced to finding a strictly proper stable  $C(s)$  to ensure that

$$(i) \quad \lambda < 1 \text{ or } \|h_3\|_{\mathcal{L}_\infty} \text{ are sufficiently small,} \quad (76)$$

$$(ii) \quad y_{des}(s) \approx D(s)r(s), \quad (77)$$

where  $D(s)$  is the desired LTI system introduced in (11). Then, Theorem 4 and Lemma 8 will imply that the output  $y(t)$  of the system in (10) and the  $\mathcal{L}_1$  adaptive control signal  $u(t)$  will follow  $y_{des}(t)$  and  $u_{des}(t)$  both in transient and steady state with quantifiable bounds, given in (58), (59) and (68)-(71).

Notice that  $\lambda < 1$  is required for stability. From (66)-(71), it follows that for achieving  $y_{des}(s) \approx D(s)r(s)$  it is desirable to ensure that  $\lambda$  or  $\|h_3\|_{\mathcal{L}_\infty}$  are sufficiently small and, in addition,  $C(s)c^\top H_o(s) \approx D(s)$ . We notice that these requirements are not in conflict with each other. So, using Lemma 2, one can consider the following conservative upper bound

$$\lambda = \|\bar{G}(s)\|_{\mathcal{L}_1} \theta_{\max} = \|H_o(s)(C(s)-1)\|_{\mathcal{L}_1} \theta_{\max} \leq \|H_o(s)\|_{\mathcal{L}_1} \|C(s)-1\|_{\mathcal{L}_1} \theta_{\max}. \quad (78)$$

Thus, minimization of  $\lambda$  can be achieved from two different perspectives: i) fix  $C(s)$  and minimize  $\|H_o(s)\|_{\mathcal{L}_1}$ , ii) fix  $H_o(s)$  and minimize the  $\mathcal{L}_1$ -gain of one of the cascaded systems  $\|H_o(s)(C(s)-1)\|_{\mathcal{L}_1}$ ,  $\|(C(s)-1)r(s)\|_{\mathcal{L}_1}$  or  $\|C(s)(C(s)-1)\|_{\mathcal{L}_1}$  via the choice of  $C(s)$ .

i) *High-gain design.* Set  $C(s) = D(s)$ . Then minimization of  $\|H_o(s)\|_{\mathcal{L}_1}$  can be achieved via high-gain feedback by choosing  $K$  sufficiently large. However, minimized  $\|H_o(s)\|_{\mathcal{L}_1}$  via large  $K$  leads to high-gain design with reduced robustness properties. Since  $C(s)$  is a strictly proper system containing the dominant poles of the closed-loop system in  $k_g c^\top H_o(s)C(s)$  and  $k_g c^\top H_o(0) = 1$ , we have  $k_g c^\top H_o(s)C(s) \approx C(s) = D(s)$ . Hence, the system response will be  $y_{ref}(s) \approx D(s)r(s)$ . We note that with large feedback  $K$ , the performance of  $\mathcal{L}_1$  adaptive controller degenerates into a high-gain type. The shortcoming of this design is that the high gain feedback  $K$  leads to a reduced phase and time-delay margin and consequently affects robustness.

ii) *Design without high-gain feedback.* As in MRAC, assume that we can select  $K$  to ensure

$$k_g c^\top H_o(s) \approx D(s). \quad (79)$$

Lemma 9: Let

$$C(s) = \frac{\omega}{s + \omega}. \quad (80)$$

For any single input  $n$ -output strictly proper stable system  $H_o(s)$  the following is true:

$$\lim_{\omega \rightarrow \infty} \|(C(s) - 1)H_o(s)\|_{\mathcal{L}_1} = 0.$$

**Proof.** It follows from (80) that  $(C(s) - 1)H_o(s) = \frac{-s}{s + \omega}H_o(s) = \frac{-1}{s + \omega}sH_o(s)$ . Since  $H_o(s)$  is strictly proper and stable,  $sH_o(s)$  is stable and has relative degree  $\geq 0$ , and hence  $\|sH_o(s)\|_{\mathcal{L}_1}$  is finite. Since  $\left\|\frac{-1}{s + \omega}\right\|_{\mathcal{L}_1} = \frac{1}{\omega}$ , it follows from (2) that  $\|(C(s) - 1)H_o(s)\|_{\mathcal{L}_1} \leq \frac{1}{\omega}\|sH_o(s)\|_{\mathcal{L}_1}$ , and the proof is complete.  $\square$

Lemma 9 states that if one chooses  $k_g c^\top H_o(s)r(s) \approx D(s)$ , then by increasing the bandwidth of the low-pass system  $C(s)$ , it is possible to render  $\|\bar{G}(s)\|_{\mathcal{L}_1}$  arbitrarily small. With large  $\omega$ , the pole  $-\omega$  due to  $C(s)$  is omitted, and  $H_o(s)$  is the dominant reference system leading to  $y_{ref}(s) \approx k_g c^\top H_o(s)r(s) \approx D(s)r(s)$ . We note that  $k_g c^\top H_o(s)$  is exactly the reference model of the MRAC design. Therefore this approach is equivalent to mimicking MRAC, and, hence, high-gain feedback can be completely avoided.

However, increasing the bandwidth of  $C(s)$  is not the only choice for minimizing  $\|\bar{G}(s)\|_{\mathcal{L}_1}$ . Since  $C(s)$  is a low-pass filter, its complementary  $1 - C(s)$  is a high-pass filter with its cutoff frequency approximating the bandwidth of  $C(s)$ . Since both  $H_o(s)$  and  $C(s)$  are strictly proper systems,  $\bar{G}(s) = H_o(s)(C(s) - 1)$  is equivalent to cascading a low-pass system  $H_o(s)$  with a high-pass system  $C(s) - 1$ . If one chooses the cut-off frequency of  $C(s) - 1$  larger than the bandwidth of  $H_o(s)$ , it ensures that  $\bar{G}(s)$  is a “no-pass” system, and hence its  $\mathcal{L}_1$  gain can be rendered arbitrarily small. This can be achieved via higher order filter design methods. The illustration is given in Fig. 4.

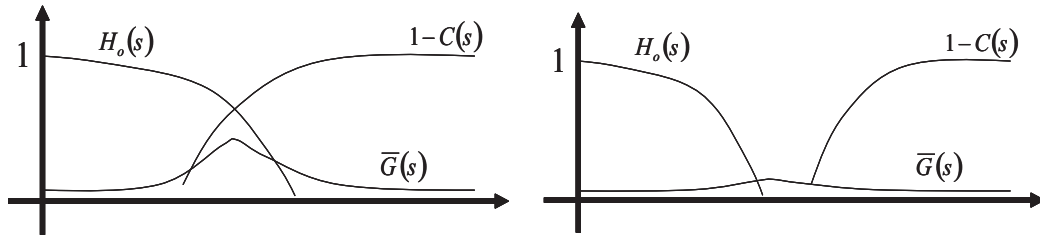


Fig. 4: Cascaded systems.

To minimize  $\|h_3\|_{\mathcal{L}_\infty}$ , we note that  $\|h_3\|_{\mathcal{L}_\infty}$  can be upperbounded in two ways:

$$(i) \quad \|h_3\|_{\mathcal{L}_\infty} \leq \|(C(s) - 1)r(s)\|_{\mathcal{L}_1} \|h_4\|_{\mathcal{L}_\infty},$$

where  $h_4(t)$  is the inverse Laplace transformation of  $H_4(s) = C(s)k_g H_o(s)\theta^\top H_o(s)$ , and

$$(ii) \quad \|h_3\|_{\mathcal{L}_\infty} \leq \|(C(s) - 1)C(s)\|_{\mathcal{L}_1} \|h_5\|_{\mathcal{L}_\infty},$$

where  $h_5(t)$  is the inverse Laplace transformation of  $H_5(s) = r(s)k_g H_o(s)\theta^\top H_o(s)$ .

We note that since  $r(t)$  is a bounded signal and  $C(s), H_o(s)$  are stable proper systems,  $\|h_4\|_{\mathcal{L}_\infty}$  and  $\|h_5\|_{\mathcal{L}_\infty}$  are finite. Therefore,  $\|h_3\|_{\mathcal{L}_\infty}$  can be minimized by minimizing  $\|(C(s) - 1)r(s)\|_{\mathcal{L}_1}$  or  $\|(C(s) - 1)C(s)\|_{\mathcal{L}_1}$ . Following the same arguments as above and assuming that  $r(t)$  is in low-frequency range, one can choose the cut-off frequency of  $C(s) - 1$  to be larger than the bandwidth of the reference signal  $r(t)$  to minimize  $\|(C(s) - 1)r(s)\|_{\mathcal{L}_1}$ . For minimization of  $\|C(s)(C(s) - 1)\|_{\mathcal{L}_1}$  notice that if  $C(s)$  is an ideal low-pass filter, then  $C(s)(C(s) - 1) = 0$  and hence  $\|h_3\|_{\mathcal{L}_\infty} = 0$ . Since an ideal low-pass filter is not physically implementable, one can minimize  $\|C(s)(C(s) - 1)\|_{\mathcal{L}_1}$  via appropriate choice of  $C(s)$ .

The above presented approaches ensure that  $C(s) \approx 1$  in the bandwidth of  $r(s)$  and  $H_o(s)$ . Therefore it follows from (66) that  $y_{des}(s) = C(s)k_g c^\top H_o(s)r(s) \approx k_g c^\top H_o(s)r(s)$ , which along with (79) yields  $y_{des}(s) \approx D(s)r(s)$ .

**Remark 4:** From Corollary 2 and Lemma 8 it follows that the  $\mathcal{L}_1$  adaptive controller can generate a system response to track (66) and (67) both in transient and steady state if we set the adaptive gain large and minimize  $\lambda$  or  $\|h_3\|_{\mathcal{L}_\infty}$ . Notice that  $u_{des}(t)$  in (67) depends upon the unknown parameter  $\theta$ , while  $y_{des}(t)$  in (66) does not. This implies that for different values of  $\theta$ , the  $\mathcal{L}_1$  adaptive controller will generate different control signals (dependent on  $\theta$ ) to ensure uniform system response (independent of  $\theta$ ). This is natural, since different unknown parameters imply different systems, and to have similar response for different systems the control signals have to be different. Here is the obvious advantage of the  $\mathcal{L}_1$  adaptive controller in a sense that it controls a partially known system as an LTI feedback controller would have done if the unknown parameters were known. Finally, we note that if the term  $k_g C(s)C(s)\theta^\top H_o(s)$  is dominated by  $k_g C(s)K^\top H_o(s)$ , then the controller in (67) turns into a robust one, and consequently the  $\mathcal{L}_1$  adaptive controller degenerates into robust design.

**Remark 5:** It follows from (63) that in the presence of large adaptive gain the  $\mathcal{L}_1$  adaptive controller and the closed-loop system state with it approximate  $u_{ref}(t), y_{ref}(t)$ . Therefore, we conclude from (38) that  $y(t)$  approximates the response of the LTI system  $c^\top (I - \bar{G}(s)\theta^\top)^{-1} G(s)$  to the input  $r(t)$ , hence its transient performance

specifications, such as overshoot and settling time, can be derived for every value of  $\theta$ . If we further minimize  $\lambda$  or  $\|h_3\|_{\mathcal{L}_\infty}$ , it follows from Lemma 8 that  $y(t)$  approximates the response of the LTI system  $C(s)c^\top H_o(s)$ . In this case, the  $\mathcal{L}_1$  adaptive controller leads to uniform transient performance of  $y(t)$  independent of the value of the unknown parameter  $\theta$ . For the resulting  $\mathcal{L}_1$  adaptive control signal one can characterize the transient specifications such as its amplitude and rate change for every  $\theta \in \Omega$ , using  $u_{des}(t)$  for it.

## 7 Discussion

### 7.1 Comparison to high-gain controller

We use a scalar system to compare the performance of the  $\mathcal{L}_1$  adaptive controller and a linear high-gain controller. Towards that end, let  $\dot{x}(t) = -\theta x(t) + u(t)$ , where  $x \in \mathbb{R}$  is the measurable system state,  $u \in \mathbb{R}$  is the control signal and  $\theta \in \mathbb{R}$  is unknown, which belongs to a given compact set  $[\theta_{\min}, \theta_{\max}]$ . Let  $u(t) = -kx(t) + kr(t)$ , leading to the following closed-loop system  $\dot{x}(t) = (-\theta - k)x(t) + kr(t)$ . We need to choose  $k > -\theta_{\min}$  to guarantee stability. We note that both the steady state error and the transient performance depend on the unknown parameter value  $\theta$ . By further introducing a proportional-integral controller, one can achieve zero steady state error. If one chooses  $k \gg \max\{|\theta_{\max}|, |\theta_{\min}|\}$ , it leads to high-gain system

$$x(s) = \frac{k}{s - (-\theta - k)}r(s) \approx \frac{k}{s + k}r(s).$$

To apply the  $\mathcal{L}_1$  adaptive controller, let the desired reference system be  $D(s) = \frac{2}{s+2}$ . Let  $u_1 = -2x$ ,  $k_g = 2$ , leading to  $H_o(s) = \frac{1}{s+2}$ . Choose  $C(s)$  as in (80) with large  $\omega_n$ , and set adaptive gain  $\Gamma_c$  large. Then it follows from Theorem 4 that

$$x(s) \approx x_{ref}(s) = C(s)k_g H_o(s)r(s) \approx \frac{\omega_n}{s + \omega_n} \frac{2}{s + 2}r(s) \approx \frac{2}{s + 2}r(s) \quad (81)$$

$$u(s) \approx u_{ref}(s) = (-2 + \theta)x_{ref}(s) + 2r(s). \quad (82)$$

The relationship in (81) implies that the control objective is met, while the relationship in (82) states that the  $\mathcal{L}_1$  adaptive controller approximates  $u_{ref}(t)$ , which cancels the unknown  $\theta$ .

### 7.2 Time-delay margin in the presence of large adaptive gain

A well-known fact in robust control is that the high gain in the feedback loop can lead to increased control effort and reduced phase margin. Since we argue that the performance bounds of  $\mathcal{L}_1$  adaptive controller can be systematically improved

by increasing the adaptation gain, in this section we provide a brief robustness analysis of the  $\mathcal{L}_1$  adaptive controller in parallel to MRAC. To enable the use of the frequency domain tools for robustness analysis, we consider a scalar linear system in the presence of constant unknown disturbance and close the loop with a MRAC controller and  $\mathcal{L}_1$  controller. So, let  $\dot{x}(t) = x(t) + u(t) + \theta$ , where  $x \in \mathbb{R}$  is the measured state,  $u \in \mathbb{R}$  is the control signal,  $\theta \in \mathbb{R}$  is an unknown constant parameter. If we apply the MRAC controller, then it reduces to the well-known PI structure:

$$u(t) = -\hat{\theta}(t) - 2x(t) + r(t), \quad \dot{x}_m(t) = -x_m(t) + r(t), \quad \dot{\hat{\theta}}(t) = \Gamma(x(t) - x_m(t)).$$

The open-loop transfer function for the time-delay margin analysis of this controller in the presence of the time-delay at the system input is  $H_o(s) = \frac{-ks+\Gamma}{s(s-1)}e^{-s\tau}$ . Application of the  $\mathcal{L}_1$  controller leads to a filtered version of the PI controller:

$$\begin{aligned} u(s) &= -C(s)(\hat{\theta}(s) - r(s)) - 2x(s), & \dot{\hat{\theta}}(t) &= -\Gamma(\hat{x}(t) - x(t)) \\ \dot{\hat{x}}(t) &= -\hat{x}(t) + u(t) + \hat{\theta}(t). \end{aligned}$$

In this case, the open-loop transfer function for the time-delay margin analysis in the presence of the time-delay at the system input is  $H_o(s) = \frac{-k}{s-1}e^{-s\tau} + \frac{\Gamma C(s)}{s^2 - a_m s + \Gamma}(e^{-s\tau} - 1)$ . We plot the time-delay margin of both systems with respect

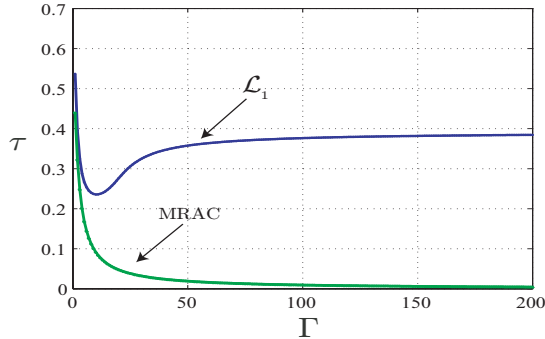


Fig. 5: Effects of adaptive gain on time-delay margin in MRAC and  $\mathcal{L}_1$  adaptive controller with  $C(s) = \frac{s}{s+1}$

to adaptive gain  $\Gamma$  in Fig. 5. We notice that the time-delay margin of PI controller goes to zero as  $\Gamma \rightarrow \infty$ , while the time-delay margin of  $\mathcal{L}_1$  adaptive control architecture is bounded away from zero as  $\Gamma \rightarrow \infty$ . Details on this analysis can be found in [22].

## 8 Time-varying unknown parameters

In this section, we consider the performance of the  $\mathcal{L}_1$  adaptive controller in the presence of time-varying unknown parameters. We prove that, in this case as well, by increasing the adaptation gain one can ensure uniform transient response for system's both signals, input and output, simultaneously. We consider the same system in (10) with unknown time-varying parameters  $\theta(t) \in \mathbb{R}^n$ , assuming that  $\theta(t) \in \Theta$ ,  $\forall t \geq 0$ . We further assume that  $\theta(t)$  is continuously differentiable with uniformly bounded derivative:

$$\|\dot{\theta}(t)\| \leq d_\theta < \infty, \quad \forall t \geq 0, \quad (83)$$

where the number  $d_\theta$  can be arbitrarily large. We consider the same reference system in (35) with  $\eta(t)$  defined as

$$\eta(t) = \theta^\top(t) x_{ref}(t). \quad (84)$$

Hence, (37) becomes

$$x_{ref}(s) = H_o(s) (k_g C(s) r(s) + (C(s) - 1) \eta(s)), \quad (85)$$

where  $\eta(s)$  is the Laplace transformation of  $\eta(t)$  in (84). Let  $\eta_1(t)$  be the signal with its Laplace transformation given by

$$\eta_1(s) = H_o(s) (C(s) - 1) \eta(s). \quad (86)$$

It can be derived easily that

$$\|\eta_1\|_{\mathcal{L}_\infty} \leq \|H_o(s)(C(s) - 1)\|_{\mathcal{L}_1} \theta_{\max} \|x_{ref}\|_{\mathcal{L}_\infty}, \quad (87)$$

where  $\theta_{\max}$  is defined in (23). It follows from Theorem 1 that the closed-loop reference system is stable if the same  $\mathcal{L}_1$ -gain requirement in (24) holds. Instead of Lemma 7 and Theorem 4, we have the following results.

**Lemma 10:** For the system in (10) in the presence of unknown time-varying  $\theta(t)$ , we have

$$\|\tilde{x}\|_{\mathcal{L}_\infty} \leq \sqrt{\theta_m / (\lambda_{\min}(P) \Gamma_c)}, \quad (88)$$

where

$$\theta_m \triangleq \max_{\theta \in \Theta} \sum_{i=1}^n 4\theta_i^2 + 2 \frac{d_\theta \lambda_{\max}(P)}{\lambda_{\min}(Q)} \max_{\theta \in \Theta} \|\theta\|. \quad (89)$$

**Proof.** Using the same candidate Lyapunov function in (25), it follows that

$$\dot{V}(t) \leq -\tilde{x}^\top(t)Q\tilde{x}(t) + 2\Gamma_c^{-1}\tilde{\theta}^\top(t)\dot{\theta}(t). \quad (90)$$

If at any  $t$ ,

$$V(t) > \theta_m/\Gamma_c, \quad (91)$$

where  $\theta_m$  is defined in (89), then it follows from (55) that

$$\tilde{x}^\top(t)P\tilde{x}(t) > 2\frac{d_\theta\lambda_{\max}(P)}{\Gamma_c\lambda_{\min}(Q)}\max_{\theta \in \Theta}\|\theta\|, \quad (92)$$

and hence

$$\tilde{x}^\top(t)Q\tilde{x}(t) > \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}\tilde{x}^\top(t)P\tilde{x}(t) > 2\frac{d_\theta\max_{\theta \in \Theta}\|\theta\|}{\Gamma_c}.$$

The upper bounds in (83) along with the projection based adaptive laws lead to the following upper bound:

$$(2\tilde{\theta}^\top(t)\dot{\theta}(t))/\Gamma_c \leq \frac{2d_\theta\max_{\theta \in \Theta}\|\theta\|}{\Gamma_c}.$$

Hence, it follows from (90) and (91) that

$$\dot{V}(t) < 0. \quad (93)$$

Since  $V(0) \leq \theta_m/\Gamma_c$ , it follows from (93) that  $V(t) \leq \theta_m/\Gamma_c$  for any  $t \geq 0$ . Since  $\lambda_{\min}(P)\|\tilde{x}(t)\|^2 \leq \tilde{x}^\top(t)P\tilde{x}(t) \leq V(t)$ , then

$$\|\tilde{x}(t)\|^2 \leq \frac{\theta_m}{\lambda_{\min}(P)\Gamma_c},$$

which concludes the proof.  $\square$

**Theorem 5:** Given the system in (10) with unknown time-varying  $\theta(t)$  and the  $\mathcal{L}_1$  adaptive controller defined via (13), (15), (16), (18) subject to (24), we have:

$$\|x - x_{ref}\|_{\mathcal{L}_\infty} \leq \gamma_3, \quad (94)$$

$$\|u - u_{ref}\|_{\mathcal{L}_\infty} \leq \gamma_4, \quad (95)$$

where

$$\gamma_3 = \frac{\|C(s)\|_{\mathcal{L}_1}}{1 - \|H_o(s)(1 - C(s))\|_{\mathcal{L}_1}\theta_{\max}}\sqrt{\frac{\theta_m}{\lambda_{\max}(P)\Gamma_c}}, \quad (96)$$

$$\gamma_4 = \left\|C(s)\frac{1}{c_o^\top H_o(s)}c_o^\top\right\|_{\mathcal{L}_1}\sqrt{\frac{\theta_m}{\lambda_{\max}(P)\Gamma_c}} + \left(\|K^\top\| + \|C(s)\|_{\mathcal{L}_1}\theta_{\max}\right) \quad (97)$$

**Proof.** Letting  $\tilde{r}(t) = \tilde{\theta}^\top(t)x(t)$ ,  $\eta_2(t) = \theta^\top(t)x(t)$ , it follows from the system in (10) and the control law in (13)-(18) that

$$x(s) = H_o(s) ((C(s) - 1)\eta_2(s) + C(s)k_g r(s) + C(s)\tilde{r}(s)) . \quad (98)$$

Following the definition of  $r_2(t)$  in (48), it follows from (85) and (98) that

$$r_2(s) = H_o(s) ((C(s) - 1)r_3(s) - C(s)\tilde{r}(s)) , \quad r_2(0) = 0 , \quad (99)$$

where  $r_3(s)$  is the Laplace transformation of the signal

$$r_3(t) = \theta^\top(t)r_2(t) . \quad (100)$$

Lemma 1 gives the following upper bound:

$$\|r_{2t}\|_{\mathcal{L}_\infty} \leq \|H_o(s)(1 - C(s))\|_{\mathcal{L}_1} \|r_{3t}\|_{\mathcal{L}_\infty} + \|r_{4t}\|_{\mathcal{L}_\infty} , \quad (101)$$

where  $r_4(t)$  is the signal with its Laplace transformation  $r_4(s) = C(s)H_o(s)\tilde{r}(s)$ . Since  $\tilde{x}(s) = H_o(s)\tilde{r}(s)$ , we have  $r_4(s) = C(s)\tilde{x}(s)$ , and hence  $\|r_{4t}\|_{\mathcal{L}_\infty} \leq \|C(s)\|_{\mathcal{L}_1} \|\tilde{x}_t\|_{\mathcal{L}_\infty}$ . Using the definition of  $\theta_{\max}$  in (23), one can easily verify from (100) that  $\|r_{3t}\|_{\mathcal{L}_\infty} \leq \theta_{\max} \|r_{2t}\|_{\mathcal{L}_\infty}$ . From (101) we have

$$\|r_{2t}\|_{\mathcal{L}_\infty} \leq \|H_o(s)(1 - C(s))\|_{\mathcal{L}_1} \theta_{\max} \|r_{2t}\|_{\mathcal{L}_\infty} + \|C(s)\|_{\mathcal{L}_1} \|\tilde{x}_t\|_{\mathcal{L}_\infty} . \quad (102)$$

The upper bound from Lemma 10 and the  $\mathcal{L}_1$ -gain requirement from (24) lead to the following upper bound

$$\|r_{2t}\|_{\mathcal{L}_\infty} \leq \frac{\|C(s)\|_{\mathcal{L}_1}}{1 - \|H_o(s)(1 - C(s))\|_{\mathcal{L}_1} \theta_{\max}} \sqrt{\frac{\theta_m}{\lambda_{\max}(P)\Gamma_c}} , \quad (103)$$

which holds uniformly for all  $t \geq 0$  and therefore leads to (94).

To prove the bound in (95), we notice that from (13)-(18) and (35)-(36) one can derive

$$u(s) - u_{ref}(s) = -K^\top(x(s) - x_{ref}(s)) + C(s)(\eta_2(s) - \eta(s)) + r_5(s) , \quad (104)$$

where  $r_5(s) = C(s)\tilde{r}(s)$ . Therefore, it follows from Lemma 1 that

$$\|u - u_{ref}\|_{\mathcal{L}_\infty} \leq \left( \|K^\top\| + \|C(s)\|_{\mathcal{L}_1} \theta_{\max} \right) \|x - x_{ref}\|_{\mathcal{L}_\infty} + \|r_5\|_{\mathcal{L}_\infty} . \quad (105)$$

We have  $r_5(s) = C(s) \frac{1}{c_o^\top H_o(s)} c_o^\top H_o(s) \tilde{r}(s) = C(s) \frac{1}{c_o^\top H_o(s)} c_o^\top \tilde{x}(s)$ , and hence,

$$\|r_5\|_{\mathcal{L}_\infty} \leq \left\| C(s) \frac{1}{c_o^\top H(s)} c_o^\top \right\|_{\mathcal{L}_1} \|\tilde{x}\|_{\mathcal{L}_\infty} .$$

Lemma 10 consequently leads to the upper bound:

$$\|r_5\|_{\mathcal{L}_\infty} \leq \left\| C(s) \frac{1}{c_o^\top H(s)} c_o^\top \right\|_{\mathcal{L}_1} \sqrt{\frac{\theta_m}{\lambda_{\max}(P)\Gamma_c}},$$

which, when substituted into (105), leads to (95).  $\square$

Since (24) ensures the stability of the reference system, it follows from Theorem 5 that the same  $\mathcal{L}_1$ -gain requirement ensures the stability of  $\mathcal{L}_1$  adaptive controller. Theorem 5 further implies that the  $\mathcal{L}_1$  adaptive controller approximates  $u_{ref}(t)$  both in transient and steady state. It is straightforward to verify that Corollary 2 holds for time-varying unknown  $\theta(t)$  as well.

We note that the control law  $u_{ref}(t)$  in the closed-loop reference system, which is used in the analysis of  $\mathcal{L}_\infty$  norm bounds, is not implementable since its definition involves the unknown parameters. So, it is important to understand how these bounds can be used for ensuring uniform transient response with *desired* specifications. We notice that the following *ideal* control signal

$$u_{ideal}(t) = k_g r(t) + \theta^\top(t) x_{ref}(t) - K^\top x_{ref}(t) \quad (106)$$

is the one that leads to desired system response:

$$\dot{x}_{ref}(t) = A_m x_{ref}(t) + b k_g r(t) \quad (107)$$

$$y_{ref}(t) = c^\top x_{ref}(t) \quad (108)$$

by cancelling the uncertainties exactly. If a part of  $u_{ideal}(t)$  is low-pass filtered by  $C(s)$  in (35), then  $u_{ref}(t)$  cancels the uncertainties dependent upon the bandwidth of  $C(s)$ . In case of fast varying  $\theta(t)$ , it is obvious that the bandwidth of the controller needs to be matched correspondingly.

## 9 Simulations

Consider the system in (10) with the following parameters:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1.4 \end{bmatrix}, \quad b = [0 \quad 1]^\top, \quad c = [1 \quad 0]^\top, \quad \theta = [4 \quad -4.5]^\top.$$

We further assume that the unknown parameter  $\theta$  belongs to a known compact set  $\Theta = \{\theta \in \mathbb{R}^2 \mid \theta_1 \in [-10, 10], \theta_2 \in [-10, 10]\}$ .

We give now the complete  $\mathcal{L}_1$  adaptive controller for this system. Since  $A$  is Hurwitz, we set  $K = 0$ . Letting  $\Gamma_c = 10000$ , we implement the  $\mathcal{L}_1$  adaptive controller following (13), (15), (16) and (18). First, we check stability of this

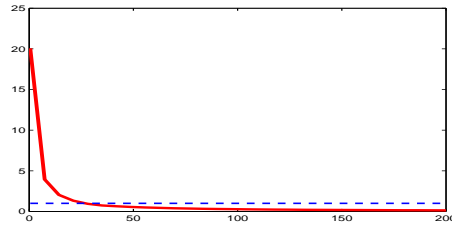
$\mathcal{L}_1$  adaptive controller. It follows from (23) that  $\theta_{\max} = 20$  and  $\|\bar{G}\|_{L_1}$  can be calculated numerically. In Fig. 6(a), we plot

$$\lambda = \|\bar{G}\|_{L_1} \theta_{\max} \quad (109)$$

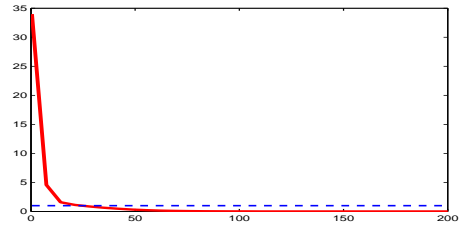
with respect to  $\omega$  and compare it to 1. We notice that for  $\omega > 30$ , we have  $\lambda < 1$ , and the  $\mathcal{L}_1$  gain requirement for stability is satisfied. So, we can choose

$$C(s) = \frac{160}{s + 160} \quad (110)$$

to ensure that  $\lambda < 0.01$ , which consequently leads to improved performance bounds in (68)-(71). For  $\omega = 160$ , we have  $\lambda = \|\bar{G}(s)\|_{L_1} \theta_{\max} = 0.1725 < 1$ , so the  $\mathcal{L}_1$ -gain requirement in (24) is indeed satisfied.



(a)  $\lambda$  (solid) defined in (109)



(b)  $\lambda$  (solid) defined in (111)

Fig. 6:  $\lambda$  (solid) with respect to  $\omega$  and constant 1 (dashed)

The simulation results of the  $\mathcal{L}_1$  adaptive controller are shown in Figs. 7(a)-7(b) for reference inputs  $r = 25, 100, 400$ , respectively. We note that it leads to scaled control inputs and scaled system outputs for scaled reference inputs. Figs. 8(a)-8(b) show the system response and the control signal for reference input  $r(t) = 100 \cos(0.2t)$ , without any retuning of the controller. Figs. 9(a)-9(b) show the system response and the control signal for reference input  $r(t) = 100 \cos(0.2t)$  and time varying  $\theta(t) = [2 + 2 \cos(0.5t) \quad 2 + 0.3 \cos(0.5t) + 0.2 \cos(t/\pi)]^\top$ , without any retuning of the controller. We note that the  $\mathcal{L}_1$  adaptive controller leads to almost identical tracking performance for both constant or time-varying unknown parameters. The control signals are different since they are adapting to different uncertainties to ensure uniform transient response.

Next, we consider a higher order filter with low adaptive gain  $\Gamma_c = 400$ ,  $C(s) = \frac{3\omega^2 s + \omega^3}{(s + \omega)^3}$ . In Fig. 6(b), we plot

$$\lambda = \|\bar{G}\|_{L_1} \theta_{\max} \quad (111)$$

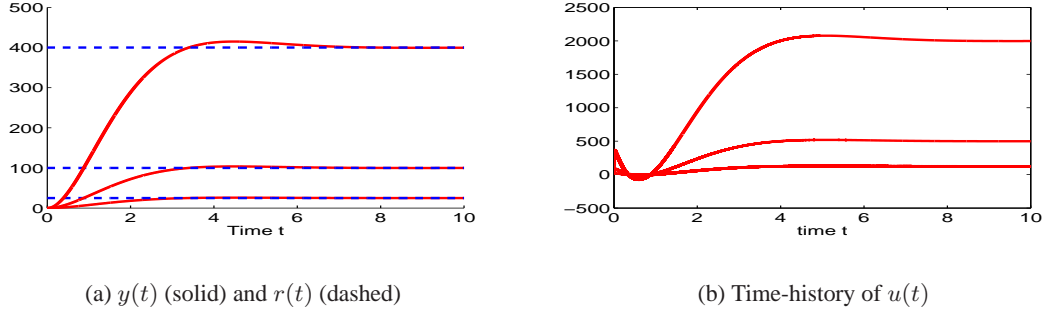


Fig. 7: Performance of  $\mathcal{L}_1$  adaptive controller with  $C(s) = \frac{160}{s+160}$  for  $r = 25, 100, 400$

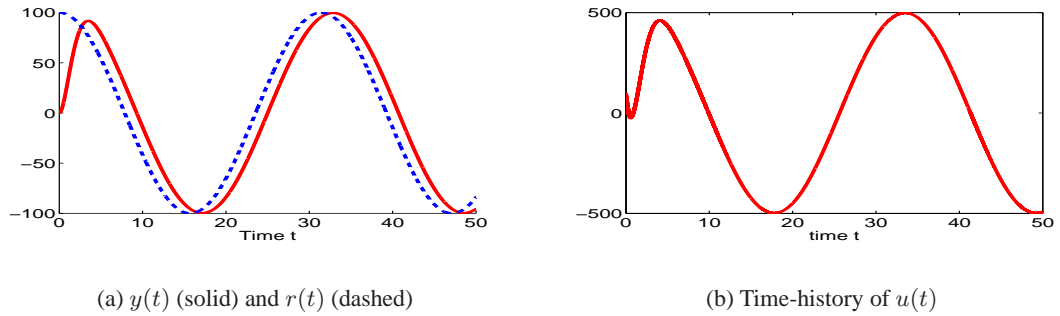


Fig. 8: Performance of  $\mathcal{L}_1$  adaptive controller with  $C(s) = \frac{160}{s+160}$  for  $r = 100 \cos(0.2t)$

with respect to  $\omega$  and compare it to 1. We notice that when  $\omega > 25$ , we have  $\lambda < 1$  and the  $\mathcal{L}_1$ -gain requirement in (24) is satisfied. Letting  $\omega = 50$  leads to  $\lambda = 0.3984$ . The simulation results of the  $\mathcal{L}_1$  adaptive controller are shown in Figs. 10(a)-10(b) for reference inputs  $r = 25, 100, 400$ , respectively. We note that it again leads to scaled control inputs and scaled system outputs for scaled reference inputs. In addition, we notice that this performance is achieved by a much smaller adaptive gain as compared to the design with the first order  $C(s)$ . Figs. 11(a)-11(b) show the system response and control signal for reference input  $r(t) = 100 \cos(0.2t)$  and time-varying  $\theta(t) = [2 + 2 \cos(0.5t) \quad 2 + 0.3 \cos(0.5t) + 0.2 \cos(t/\pi)]^\top$ , without any retuning of the controller.

**Remark 6:** The simulations pointed out that with higher order filter  $C(s)$  one

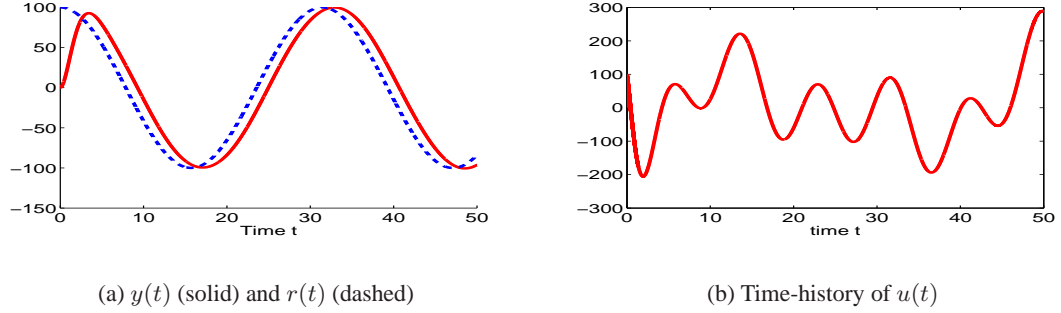


Fig. 9: Performance of  $\mathcal{L}_1$  adaptive controller with  $C(s) = \frac{160}{s+160}$  for  $r = 100 \cos(0.2t)$  with time-varying  $\theta(t) = [2 + 2 \cos(0.5t) \ 2 + 0.3 \cos(0.5t) + 0.2 \cos(t/\pi)]^\top$

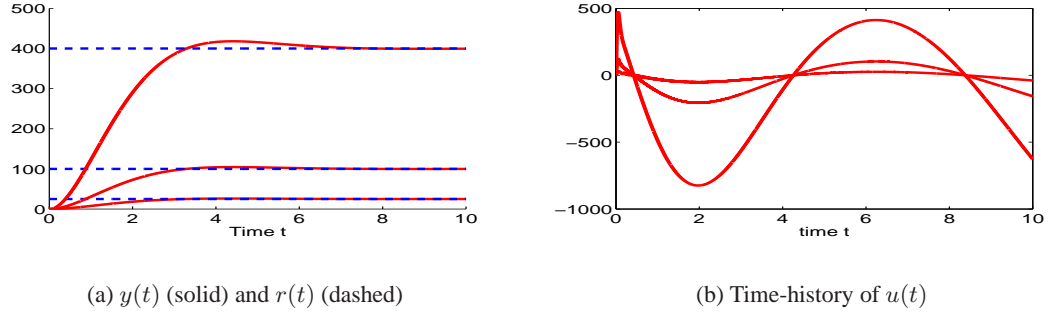


Fig. 10: Performance of  $\mathcal{L}_1$  adaptive controller with  $C(s) = \frac{7500s+50^3}{(s+50)^3}$  for  $r = 25, 100, 400$

could use relatively small adaptive gain. While a rigorous relationship between the choice of the adaptive gain and the order of the filter cannot be derived, an insight into this can be gained from the following analysis. It follows from (10), (13) and (18) that  $x(s) = G(s)r(s) + H_o(s)\theta^\top x(s) + H_o(s)C(s)\bar{r}(s)$ , while the state predictor in (20) can be rewritten as  $\hat{x}(s) = G(s)r(s) + H_o(s)(C(s) - 1)\bar{r}(s)$ . We note that  $\bar{r}(t)$  is divided into two parts. Its low-frequency component  $C(s)\bar{r}(s)$  is what the system gets, while the complementary high-frequency component  $(C(s) - 1)\bar{r}(s)$  goes into the state predictor. If the bandwidth of  $C(s)$  is large, then it can suppress only the high frequencies in  $\bar{r}(t)$ , which appear only in the presence of large adaptive gain. A properly designed higher order  $C(s)$  can be more effective to serve the

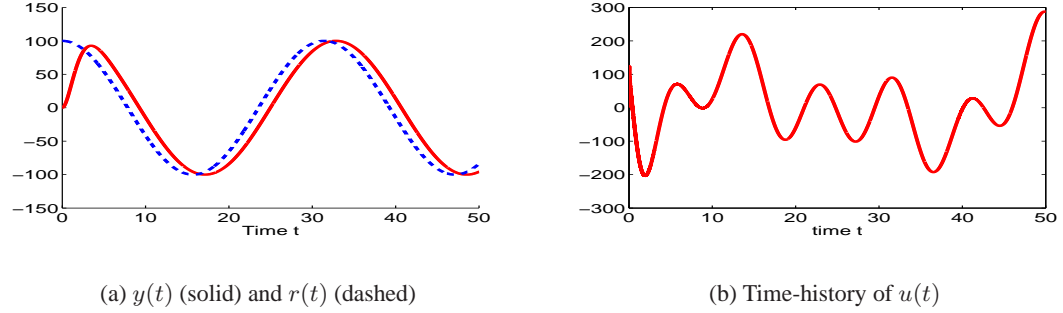


Fig. 11: Performance of  $\mathcal{L}_1$  adaptive controller with  $C(s) = \frac{7500s+50^3}{(s+50)^3}$  for  $r = 100 \cos(0.2t)$  with time-varying  $\theta(t) = [2 + 2 \cos(0.5t) \quad 2 + 0.3 \cos(0.5t) + 0.2 \cos(t/\pi)]^\top$

purpose of filtering with reduced tailing effects, and, hence can generate similar  $\lambda$  with smaller bandwidth. This further implies that similar performance can be achieved with smaller adaptive gain.

The  $\mathcal{L}_1$  adaptive controller has been successfully flight tested on a miniature aerial vehicle (UAV) with limited payload capabilities, which consequently restricted the increase of the adaptation rate [23]. Nevertheless, the flight tests verified that  $\mathcal{L}_1$  adaptive controller did not require any tuning. Refs. [24, 25] report application of  $\mathcal{L}_1$  controller to different aerospace benchmark problems.

## 10 Conclusion

A novel adaptive control architecture is presented that has guaranteed transient response in addition to stable tracking. The new low-pass control architecture adapts fast without generating high-frequency oscillations in the control signal and leads to scaled response for both system's input and output signals, which is otherwise not possible to achieve using conventional adaptive controllers. The low-frequency behavior of the control signal implies that the rate saturation is no more an issue. These arguments enable development of theoretically justified tools for verification and validation of adaptive controllers. Extension of the methodology to systems with unknown high frequency gain will be reported in an upcoming publication.

## References

- [1] K. S. Narendra and A. M. Annaswamy. *Stable Adaptive Systems*. Prentice-Hall, Inc., 1989.
- [2] J.-J. E. Slotine and W. Li. *Applied Nonlinear Control*. Prentice Hall, Englewood Cliffs, NJ, 1991.
- [3] A. Datta and M.-T. Ho. On modifying model reference adaptive control schemes for performance improvement. *IEEE Trans. Autom. Contr.*, 39(9):1977–1980, September 1994.
- [4] G. Bartolini, A. Ferrara, and A. A. Stotsky. Robustness and performance of an indirect adaptive control scheme in presence of bounded disturbances. *IEEE Trans. Autom. Contr.*, 44(4):789–793, April 1999.
- [5] J. Sun. A modified model reference adaptive control scheme for improved transient performance. *IEEE Trans. Autom. Contr.*, 38(7):1255–1259, July 1993.
- [6] D.E. Miller and E.J. Davison. Adaptive control which provides an arbitrarily good transient and steady-state response. *IEEE Trans. Autom. Contr.*, 36(1):68–81, January 1991.
- [7] R. Costa. Improving transient behavior of model-reference adaptive control. *Proc. of American Control Conference*, pages 576–580, 1999.
- [8] P. Ioannou and J. Sun. *Robust Adaptive Control*. Prentice Hall, 1996.
- [9] B.E. Ydstie. Transient performance and robustness of direct adaptive control. *IEEE Trans. Autom. Contr.*, 37(8):1091–1105, August 1992.
- [10] M. Krstic, P. V. Kokotovic, and I. Kanellakopoulos. Transient performance improvement with a new class of adaptive controllers. *Systems & Control Letters*, 21:451–461, 1993.
- [11] R. Ortega. Morse’s new adaptive controller: Parameter convergence and transient performance. *IEEE Trans. Autom. Contr.*, 38(8):1191–1202, August 1993.
- [12] Z. Zang and R. Bitmead. Transient bounds for adaptive control systems. *Proc. of 30<sup>th</sup> IEEE Conference on Decision and Control*, pages 2724–2729, December 1990.

- [13] A. Datta and P. Ioannou. Performance analysis and improvement in model reference adaptive control. *IEEE Trans. Autom. Contr.*, 39(12):2370–2387, December 1994.
- [14] A. M. Arteaga and Y. Tang. Adaptive control of robots with an improved transient performance. *IEEE Trans. Autom. Contr.*, 47(7):1198–1202, July 2002.
- [15] K. S. Narendra and J. Balakrishnan. Improving transient response of adaptive control systems using multiple models and switching. *IEEE Trans. Autom. Contr.*, 39(9):1861–1866, September 1994.
- [16] P. Zigang and T. Basar. Adaptive controller design for tracking and disturbance attenuation in parametric strict-feedback nonlinear systems. *IEEE Trans. Autom. Contr.*, 43(8):1066–1083, August 2005.
- [17] G. Arslan and T. Basar. Disturbance attenuating controller design for strict-feedback systems with structurally unknown dynamics. *Automatica*, 37:1175–1188, 2005.
- [18] Y. Zhang and P. Ioannou. A new linear adaptive controller: Design, analysis and performance. *IEEE Trans. Autom. Contr.*, 45(5):883–897, May 2000.
- [19] M. Krstic, I. Kanellakopoulos, and P. Kokotovic. *Nonlinear and Adaptive Control Design*. John Wiley & Sons, New York, 1995.
- [20] H. K. Khalil. *Nonlinear Systems*. Prentice Hall, Englewood Cliffs, NJ, 2002.
- [21] K. Zhou and J. C. Doyle. *Essentials of Robust Control*. Prentice Hall, Englewood Cliffs, NJ, 1998.
- [22] C. Cao, V. V. Patel, K. Reddy, N. Hovakimyan, E. Lavretsky, and K. Wise. Are the phase and time-delay margins always adversely affected by high-gain? *In Proc. of AIAA Guidance, Navigation and Control Conference*, 2006.
- [23] R. W. Beard, N. Knoebel, C. Cao, N. Hovakimyan, and J. Matthews. An  $\mathcal{L}_1$  adaptive pitch controller for miniature air vehicles. *In Proc. of AIAA Guidance, Navigation and Control Conference*, 2006.
- [24] C. Cao, N. Hovakimyan, and E. Lavretsky. Application of  $\mathcal{L}_1$  adaptive controller to wing rock. *In Proc. of AIAA Guidance, Navigation and Control Conference*, 2006.

- 
- [25] J. Wang, C. Cao, V. Patel, N. Hovakimyan, and E. Lavretsky.  $\mathcal{L}_1$  adaptive neural network controller for autonomous aerial refueling with guaranteed transient performance. *In Proc. of AIAA Guidance, Navigation and Control Conference*, 2006.